Questions and Answers III

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Q: can you give an example of wall-crossing?

A: here's one flavor of examples. Consider a particular $N = 2$ SUSY theory, Argyres-Douglas theory, which takes as input a pair of ADE Dynkin diagrams. The simplest one is (A_1, A_2). The corresponding moduli space M is Higgs bundles (for either A_1 or A_2, as it turns out) on $\mathbb{CP}^1$ with one irregular singularity. The Hitchin base B is quadratic differentials on $\mathbb{C}$ of the form

$$\varphi_2(z) = z^3 + z + u \tag{1}$$

where u is a parameter. The Coulomb branch is the u plane. The spectral curve is

$$y^2 = z^3 + z + u \tag{2}$$

and the singular points are the points where the discriminant of the RHS is zero. There are two of these. Wall-crossing occurs for quantities $\Omega(\gamma, u)$ where $\gamma \in H_1(\Sigma_u, \mathbb{Z})$. It turns out to be the number of saddle connections of the quadratic differential $\varphi_2(z)$ with charge γ. Here a saddle connection is a geodesic of the metric $\sqrt{|\varphi_2|^2}$ with connects zeroes of the quadratic differential. For u small there are three zeroes and two saddle connections. Each of these lifts up to a loop on the spectral curve. This corresponds to $\Omega(\pm\gamma_1) = 1, \Omega(\pm\gamma_2) = 1$. For u sufficiently large the three zeroes form a triangle and there are three saddle connections. This corresponds to $\Omega(\gamma_1) = 1, \Omega(\gamma_2) = 1, \Omega(\gamma_1 + \gamma_2) = 1$.

The wall is determined by the condition that the ratio of two hypergeometric functions is real. To see this we use the extra fact that each saddle connection has a phase ϑ such that $\sqrt{\varphi_2}e^{i\theta}$ is real along the geodesic. If the saddle connection has charge γ then ϑ is the argument of Z_γ. The trajectories with a given phase foliate $\mathbb{C}$.

(This explanation wasn't completed and another question was asked that I didn't catch.)